A note on fluctuating heat transfer at small Péclet numbers

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1. Introduction

The hot-wire anemometer, used for recording speed variations in turbulent flow, involves in its working principle the unsteady heat transfer from a hot fixed surface to a fluctuating air stream moving past the surface. If the wire is maintained at a constant (high) temperature, the rate of loss of heat from the wire changes with the velocity of the incident stream, and the compensating rate of gain of heat, produced by the Joule heating effect of the electric current, changes correspondingly. The accompanying change of current can be measured, and used to calculate the varying velocity of the air stream. The hot wire may have a diameter as low as 10^{-4} in. and the Reynolds number of the flow is then of the order of 0·05 for each ft. per sec of velocity. With low velocities, of the order of 10 or 20 ft./sec, the flow past the wire is in the range of small Reynolds number, and the exact equations of flow may be approximated by simpler equations in the manner of Oseen's theory (Lamb 1932). The approximate equations are not easy to solve when the flow is compressible, as it will be in the presence of the large temperature differences imposed by the heat of the wire. If, however, the temperature differences are assumed to be small, the approximate energy equation is no longer linked with the equations of continuity and momentum, and it may be solved without knowledge of the velocity field. The purpose of this note is to give the solution for the temperature field when a warm circular cylinder or a warm sphere is held at rest in a fluctuating stream.

2. The temperature equation

The energy equation from which we start may be written as

$$\rho \frac{D}{Dt}(c_p T) = \operatorname{div}(\lambda \operatorname{grad} T), \tag{1}$$

where t is the time, ρ the density, T the temperature, c_p the specific heat at constant pressure, and λ the thermal conductivity. This equation differs from the exact energy equation, for a compressible fluid, in the omission of the rate of working of the pressure forces and the rate of heat production through the action of viscosity. The weights of these omitted terms are in the ratio $M^2 : (\chi - 1)$ compared with the terms retained, where M is the Mach number of the free stream and χ is T_w/T_∞, the ratio of the temperatures of the obstacle and the free stream. In quoting (1) we have assumed that $M^2 \ll (\chi - 1)$, and we now go further and assume also that $\chi - 1 \ll 1$ so that temperature differences in the flow are small. This allows us to neglect density variations and to regard λ as a constant, so that (1) becomes

$$\frac{DT}{Dt} = \kappa \nabla^2 T, \tag{2}$$

where κ is the thermometric conductivity. If the stream is slow moving, (2) can be replaced by the Oseen-type equation

$$\left(\frac{\partial}{\partial t} + U_\infty \frac{\partial}{\partial x}\right) T = \kappa \nabla^2 T, \tag{3}$$

in which U_∞ is the velocity of the free stream and x is the Cartesian co-ordinate in the direction of U_∞. In this equation, the convection term $\mathbf{v} . \operatorname{grad} T$ from (2) has been replaced by its form at infinity; if it is omitted altogether, on the grounds that $\mathbf{v}$ and grad T are both small, the Stokes-type equation

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T \tag{4}$$

is obtained, but this yields a solution for a circular cylinder that cannot satisfy the boundary condition at infinity.

We may notice at this point that if we introduce non-dimensional variables

$$\tilde{t} = \omega t, \quad \tilde{x} = Px/l, \tag{5}$$

in which ω is a representative frequency in the fluctuating motion, l is a representative length, and P is the Péclet number Ul/κ, where U is a representative velocity, then (3) becomes

$$\frac{\omega\kappa}{U^2}\frac{\partial T}{\partial \tilde{t}} + \frac{U_\infty}{U}\frac{\partial T}{\partial \tilde{x}} = \tilde{\nabla}^2 T. \tag{6}$$

This shows that (3) stands as the appropriate form of the energy equation provided $\omega\kappa/U^2$ is $O(1)$, but that if ω is very small the term $\partial T/\partial t$ can be omitted (the quasi-steady case), and if ω is very large the term $U_\infty \partial T/\partial x$ is negligible so that the equation reduces to Stokes's form. If we denote the Reynolds number Ul/ν by R, and the Prandtl number by σ, the Péclet number P is σR. For air, with $\sigma = 0{\cdot}72$, smallness of the Reynolds number implies smallness of the Péclet number, but, as (5) and (6) indicate, it is the Péclet number and not the Reynolds number which is the fundamental parameter in this problem.

We shall now assume that the free stream is fluctuating in simple harmonic motion about a mean value U with a small amplitude and an angular frequency ω, so that

$$U_\infty = U(1 + \epsilon\, e^{i\omega t}), \tag{7}$$

where $\epsilon \ll 1$. If the temperature is expressed in terms of a function f by the relation

$$T = T_\infty[1 + (\chi - 1)f], \tag{8}$$

equation (3) becomes

$$\frac{\partial f}{\partial t} + U(1 + \epsilon\, e^{i\omega t})\frac{\partial f}{\partial x} = \kappa \nabla^2 f, \tag{9}$$

and f must satisfy the boundary conditions: $f = 1$ on the obstacle, $f \to 0$ at infinity.

It is appropriate to write
$$f = f_0 + \epsilon\, e^{i\omega t} f_1, \tag{10}$$
where f_0, f_1 satisfy
$$\left(\nabla^2 - 2k\frac{\partial}{\partial x}\right) f_0 = 0, \tag{11}$$
$$\left(\nabla^2 - 2k\frac{\partial}{\partial x} - \frac{i\omega}{\kappa}\right) f_1 = 2k\frac{\partial f_0}{\partial x}, \tag{12}$$
in which $k = U/2\kappa$. Finally, with the substitutions
$$f_0 = e^{kx} g_0, \quad f_1 = e^{kx} g_1,$$
equations (11) and (12) reduce to
$$(\nabla^2 - k^2)\, g_0 = 0, \tag{13}$$
$$[\nabla^2 - (k^2 + i\omega/\kappa)]\, g_1 = 2k\left(\frac{\partial g_0}{\partial x} + k g_0\right), \tag{14}$$
with the boundary conditions:
$$g_0 = e^{-kx}, \quad g_1 = 0, \quad \text{on the obstacle}, \tag{15}$$
$$g_0 \to 0, \quad g_1 \to 0, \quad \text{at infinity}. \tag{16}$$

3. Circular cylinder

In plane polar co-ordinates r, θ, the appropriate solution of (13) is
$$g_0 = \sum_{m=0}^{\infty} a_m K_m(s) \cos m\theta, \tag{17}$$
where the a_m are constants to be determined, K_m is a Bessel function in the usual notation (Watson 1944) and $s = kr$. The function g_0 must be a single-valued even function of θ that vanishes for large r. From the boundary condition (15),
$$\sum_{m=0}^{\infty} a_m K_m(s_0) \cos m\theta = e^{-s_0 \cos\theta}, \tag{18}$$
where $s_0 = ka$, and a is the radius of the cylinder; and since
$$e^{-s\cos\theta} = I_0(s) + 2\sum_{m=0}^{\infty} (-1)^m I_m(s) \cos m\theta,$$
it follows that
$$a_m = \begin{cases} I_0(s_0)/K_0(s_0) & (m = 0), \\ 2(-1)^m I_m(s_0)/K_m(s_0) & (m \geqslant 1). \end{cases} \tag{19}$$

Equation (14) in polar co-ordinates is
$$(\nabla^2 - \alpha^2 k^2)\, g_1 = 2k^2 \left[\cos\theta \frac{\partial g_0}{\partial s} - \frac{\sin\theta}{s}\frac{\partial g_0}{\partial \theta} + g_0\right]$$
$$= 2k^2 \sum_{m=0}^{\infty} A_m K_m(s) \cos m\theta, \tag{20}$$
where
$$A_m = \begin{cases} \dfrac{1}{s_0 K_0(s_0) K_1(s_0)} & (m = 0), \\ \dfrac{2(-1)^{m+1} K'_m(s_0)}{s_0 K_{m-1}(s_0) K_m(s_0) K_{m+1}(s_0)} & (m \geqslant 1). \end{cases} \tag{21}$$
Also
$$\alpha^2 = 1 + \frac{i\omega}{k^2\kappa} = 1 + \frac{4ih}{P}, \tag{22}$$

in which h is the frequency parameter $\omega l/U$; in the present case, $l = 2a$. The required solution of (20), vanishing at $s = s_0$ and $s = \infty$, is

$$g_1 = \frac{iP}{2h}\sum_{m=0}^{\infty} A_m\left[K_m(s) - \frac{K_m(s_0)}{K_m(\alpha s_0)}K_m(\alpha s)\right]\cos m\theta. \tag{23}$$

Since $$e^{s\cos\theta}\cos m\theta = I_m(s) + \sum_{n=1}^{\infty}[I_{m-n}(s) + I_{m+n}(s)]\cos n\theta,$$

we obtain $$f_0 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_m F_{mn}(s,s)\cos n\theta, \tag{24}$$

where $$F_{mn}(\xi,\eta) = \begin{cases} K_m(\xi)\, I_m(\eta) & (n=0), \\ K_m(\xi)\,[I_{m-n}(\eta) + I_{m+n}(\eta)] & (n \geqslant 1), \end{cases} \tag{25}$$

and $$f_1 = \frac{iP}{2h}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_m\left[F_{mn}(s,s) - \frac{K_m(s_0)}{K_m(\alpha s_0)}F_{mn}(\alpha s, s)\right]\cos n\theta. \tag{26}$$

This completes the required solution for the temperature distribution.

It remains to determine the fluctuating heat transfer from the cylinder to the stream. The total heat flux per unit length from the cylinder is

$$Q = -\int_0^{2\pi}\lambda\left(\frac{\partial T}{\partial r}\right)_{r=a} a\,d\theta,$$

and the corresponding Nusselt number, defined as

$$C_q = \frac{Q}{2\pi a[\lambda(T_w - T_\infty)/2a]},$$

is given by $$C_q = -\frac{ka}{\pi}\int_0^{2\pi}\left(\frac{\partial f}{\partial s}\right)_{s=s_0} d\theta.$$

It follows that $$C_q = C_{q0} + \epsilon\, e^{i\omega t}C_{q1}, \tag{27}$$

where $$C_{q0} = 2\left[\frac{I_0(s_0)}{K_0(s_0)} + 2\sum_{m=1}^{\infty}(-1)^m\frac{I_m(s_0)}{K_m(s_0)}\right], \tag{28}$$

and $$C_{q1} = -\frac{iP^2}{4h}\sum_{m=0}^{\infty} A_m K_m(s_0)\, I_m(s_0)\left[\frac{K'_m(s_0)}{K_m(s_0)} - \frac{\alpha K'_m(\alpha s_0)}{K_m(\alpha s_0)}\right]. \tag{29}$$

These expressions are, however, in a more general form than the Oseen approximation warrants, for the approximation is only valid if P, which is equal to $4s_0$, is small. Accordingly, we replace the Bessel functions by their series expansions for small values of the argument. If we write

$$\beta = \frac{P}{8} \quad \text{and} \quad L(\beta) = [\ln(\beta^{-1}) - \gamma]^{-1}, \tag{30}$$

where γ is Euler's constant, we obtain from (21)

$$\begin{aligned} A_0 &= L(\beta) + [2 - L^2(\beta)]\beta^2 + O(\beta^4), \\ A_1 &= -L(\beta) - [4 - L^2(\beta)]\beta^2 + O(\beta^4), \\ A_2 &= 2\beta^2 + O(\beta^4), \\ A_n &= O(\beta^{2n-2}) \quad \text{for} \quad n \geqslant 3. \end{aligned}$$

It follows from (28) and (29), when terms $O(\beta^4)$ are neglected, that

$$C_{q0} = 2\{L(\beta) - [4 + L^2(\beta)]\beta^2\}, \tag{31}$$

$$C_{q1} = i\frac{8\beta}{h}\left\{L(\beta) - L(\alpha\beta) + \left[\alpha^2 L^2(\alpha\beta) - L^2(\beta) + 2\left(\alpha^2\frac{L(\beta)}{L(\alpha\beta)} - \frac{L(\alpha\beta)}{L(\beta)}\right) + \frac{ih}{\beta}(L(\alpha\beta)+1)\right]\beta^2\right\}. \tag{32}$$

In quoting these formulae we must remember that they would be modified if the second approximation to the convection term $\mathbf{v}.\text{grad}\, T$ in (2) were used instead of the first approximation $U_\infty(\partial/\partial x)$ shown in (3). To introduce the second approximation would require the calculation of the fluctuating velocity field according to Oseen's equations. Recent work improving the Oseen theory for the velocity field in steady flow past a circular cylinder (Kaplun 1957) suggests that the leading term, $2L(\beta)$, in C_{q0} would be modified, although only slightly for small values of β, in the next approximation. The effect on the terms in β^2 is likely to be considerable, so we shall confine attention here to the leading terms in the expressions for C_{q0} and C_{q1}. Actually the term involving β^2 in (31) is only about 7 % of the leading term for a Reynolds number of 1 in air ($\beta = 0{\cdot}09$), and it is an even smaller fraction for smaller Reynolds numbers. It is therefore certainly permissible to concentrate on the leading terms.

$\beta(=\frac{1}{8}P)$	0·01	0·02	0·04	0·06	0·08	0·10
C^*_{q1}	0·123	0·180	0·287	0·400	0·527	0·672
β	0·12	0·14	0·16	0·18	0·20	—
C^*_{q1}	0·840	1·037	1·269	1·545	1·877	—

TABLE 1

$4h/P$	0·1		0·2		0·4		0·6		0·8		1·0	
β	m	$\delta°$	m	$\delta°$	m	$\delta°$	m	$\delta°$	m	$\delta°$	m	$\delta°$
0·01	0·998	2·15	0·994	4·27	0·977	8·31	0·952	11·97	0·923	15·20	0·892	18·00
0·02	0·999	2·00	0·994	3·97	0·978	7·73	0·955	11·14	0·928	14·13	0·899	16·74
0·04	0·999	1·78	0·995	3·53	0·980	6·86	0·959	9·87	0·934	12·50	0·907	14·77
0·06	0·999	1·58	0·995	3·14	0·982	6·09	0·962	8·75	0·939	11·06	0·914	13·03
0·08	0·999	1·39	0·996	2·76	0·983	5·38	0·965	7·67	0·943	9·66	0·920	11·33
0·10	0·999	1·20	0·996	2·38	0·984	4·61	0·967	6·57	0·947	8·24	0·925	9·62
0·12	0·999	1·01	0·996	1·99	0·985	3·84	0·969	5·44	0·950	6·76	0·930	7·81
0·14	0·999	0·80	0·996	1·58	0·986	3·02	0·971	4·24	0·953	5·20	0·933	5·90
0·16	0·999	0·58	0·996	1·14	0·987	2·16	0·972	2·97	0·955	3·53	0·936	3·86
0·18	0·999	0·34	0·997	0·67	0·987	1·23	0·973	1·59	0·956	1·73	0·938	1·65
0·20	0·999	0·09	0·997	0·16	0·987	0·21	0·974	0·09	0·957	−0·24	0·938	−0·76

It will be noted that there is a phase lag of the heat transfer behind the velocity fluctuation, except at the two highest frequencies (in the last two columns) for the highest Péclet number quoted, for which there is a phase advance.

TABLE 2

It will be convenient to introduce the quasi-steady value of the expression

$$C_{q1} = i\frac{8\beta}{h}[L(\beta) - L(\alpha\beta)],$$

obtained by letting the frequency tend to zero ($h \to 0$ and $\alpha \to 1$). This is given by

$$C^*_{q1} = 2L^2(\beta), \tag{33}$$

and we shall then write

$$\frac{C_{q1}}{C^*_{q1}} = i\frac{4\beta}{h}\frac{L(\beta)-L(\alpha\beta)}{L^2(\beta)} = m\,e^{-i\delta}, \tag{34}$$

where m is the magnification factor and δ the phase lag of the fluctuating component of the heat transfer compared with its quasi-steady value. Values of C^*_{q1} for a range of Péclet numbers are given in Table 1, and the values of m amd δ for various frequencies and for the same range of Péclet numbers are given in Table 2.

4. Sphere

Although it is not relevant to the hot-wire anemometer, the corresponding problem of fluctuating heat transfer from a warm sphere will now be briefly considered. With spherical polar co-ordinates r, θ, λ, there is no dependence on the azimuthal angle λ because of axial symmetry, and the required solution of (13) may be written as

$$g_0 = \sum_{m=0}^{\infty} a_m \chi_m(s)\, P_m(\cos\theta), \tag{35}$$

where the a_m are constants to be determined,

$$\chi_m(s) = (2m+1)\,(\pi/2s)^{\frac{1}{2}}\, K_{m+\frac{1}{2}}(s), \tag{36}$$

and P_m denotes the Legrendre polynomial of degree m. Since

$$e^{-s\cos\theta} = \sum_{m=0}^{\infty} (-1)^m\,(2m+1)\,(\pi/2s)^{\frac{1}{2}}\, I_{m+\frac{1}{2}}(s)\, P_m(\cos\theta),$$

the boundary condition corresponding to (18) gives

$$a_m = (-1)^m \frac{I_{m+\frac{1}{2}}(s_0)}{K_{m+\frac{1}{2}}(s_0)}. \tag{37}$$

The equation corresponding to (20) is

$$(\nabla^2 - \alpha^2 k^2)\, g_1 = 2k^2 \sum_{m=0}^{\infty} A_m \chi_m(s)\, P_m(\cos\theta), \tag{38}$$

where

$$A_m = (-1)^{m+1}\left[K'_{m+\frac{1}{2}}(s_0) + \frac{K_{m+\frac{1}{2}}(s_0)}{2s_0}\right][s_0 K_{m-\frac{1}{2}}(s_0)\, K_{m+\frac{1}{2}}(s_0)\, K_{m+\frac{3}{2}}(s_0)]^{-1}.$$

The required solution of (38), vanishing at the surface of the sphere and at infinity, is

$$g_1 = \frac{iP}{2h}\sum_{m=0}^{\infty} A_m\left[\chi_m(s) - \frac{\chi_m(s_0)}{\chi_m(\alpha s_0)}\,\chi_m(\alpha s)\right] P_m(\cos\theta). \tag{39}$$

Since

$$e^{s\cos\theta} P_m(\cos\theta) = \sum_{n=0}^{\infty} e_{mn}(s)\, P_n(\cos\theta),$$

where

$$e_{mn}(s) = (2n+1)\sum_{r=0}^{\infty}\frac{(2r)!\,(2m-2r)!\,(2n-2r)!}{(r!)^2\,(2m+2n-2r+1)!}\left[\frac{(m+n-r)!}{(m-r)!\,(n-r)!}\right]^2 \times (2m+2n-4r+1)\,(\pi/2s)^{\frac{1}{2}}\, I_{m+n-2r+\frac{1}{2}}(s),$$

it follows that
$$f_0 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_m F_{mn}(s,s)\,P_n(\cos\theta),$$
where
$$F_{mn}(\xi,\eta) = \chi_m(\xi)\,e_{mn}(\eta), \tag{40}$$
and
$$f_1 = \frac{iP}{2h}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_m\left[F_{mn}(s,s) - \frac{\chi_m(s_0)}{\chi_m(\alpha s_0)}F_{mn}(\alpha s,s)\right]P_n(\cos\theta). \tag{41}$$

The heat flux from the surface of the sphere is
$$Q = -\int_0^{\pi}\lambda\left(\frac{\partial T}{\partial r}\right)_{r=a} 2\pi a^2\sin\theta\,d\theta$$
and the Nusselt number, defined as
$$C_q = \frac{Q}{4\pi a^2[\lambda(T_w - T_\infty)/2a]},$$
is given by
$$C_q = -ka\int_{-1}^{1}\left(\frac{\partial f}{\partial s}\right)_{s=s_0} d(\cos\theta).$$

$4h/P$	0·1	0·2	0·4	0·6	0·8	1·0
m	0·999	0·996	0·986	0·970	0·952	0·931
$\delta°$	1·43	2·85	5·59	8·15	10·49	12·62

TABLE 3

Hence
$$C_q = C_{q0} + \epsilon\,e^{i\omega t}C_{q1}, \tag{42}$$
where
$$C_{q0} = \frac{\pi}{s_0}\sum_{m=0}^{\infty}(2m+1)(-1)^m\frac{I_{m+\frac{1}{2}}(s_0)}{K_{m+\frac{1}{2}}(s_0)}, \tag{43}$$
and
$$C_{q1} = -\frac{i\pi P}{2h}\sum_{m=0}^{\infty}(2m+1)A_m I_{m+\frac{1}{2}}(s_0)K_{m+\frac{1}{2}}(s_0)\left[\frac{K'_{m+\frac{1}{2}}(s_0)}{K_{m+\frac{1}{2}}(s_0)} - \frac{\alpha K'_{m+\frac{1}{2}}(\alpha s_0)}{K_{m+\frac{1}{2}}(\alpha s_0)}\right]. \tag{44}$$

As before, we must restrict attention to small values of the arguments of the Bessel functions, and we obtain
$$C_{q0} = 2 + 4\beta + O(\beta^2), \tag{45}$$
$$C_{q1} = -\frac{2iP}{h}(\alpha - 1)\beta + O(\beta^2). \tag{46}$$

We have not included terms $O(\beta^2)$ because they are likely to be modified when the neglected convection terms are brought back into the energy equation. The quasi-steady value of C_{q1} is
$$C^*_{q1} = 4\beta,$$
and so
$$\frac{C_{q1}}{C^*_{q1}} = m\,e^{-i\delta} = -\frac{iP}{2h}(\alpha - 1).$$
This is a function of h/P only, and the values of m and δ for the same range of values of h/P as were used for the circular cylinder are given in Table 3.

REFERENCES

KAPLUN, S. 1957 *J. Math. Mech.* **6**, 595–603.

LAMB, SIR HORACE 1932 *Hydrodynamics*, 6th edition. Cambridge University Press.

WATSON, G. N. 1944 *Theory of Bessel Functions*, 2nd edition. Cambridge University Press.